BPX PRECONDITIONER FOR MNNV571

$\Omega \subset \mathbb{R}^2$ polygonal domain, conforming triangulation, P^1 discretization
sequence of nested spaces associated with uniform refinement

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M}_J =: \mathcal{M} \qquad (J \geq 2)$$

solving the problem: find $U \in \mathcal{M}$ such that

$$A(U, \phi) = (f, \phi) \qquad \forall \phi \in \mathcal{M}$$

A is symmetric, self-adjoint and elliptic form

The goal: find a preconditioner $\mathcal{B}$ such that
(1) the action of $\mathcal{B}$ on vectors from $\mathcal{M}$ is economical to compute
(2) $\mathcal{K}(\mathcal{B}A)$ is not too large

$\mathcal{K}(\mathcal{B}A) \leq c_1/c_0$ for any $c_0, c_1 > 0$ satisfying

$$c_0 \left(A(v,v) \right) \leq A(\mathcal{B}Av, v) \leq c_1 A(v,v) \qquad \forall v \in \mathcal{M}$$

Define the operators, for $k = 1, \dots, J$

- $P_k : \mathcal{M} \to \mathcal{M}_k$
$$A(P_k w, v_k) = A(w, v_k) \qquad \forall v_k \in \mathcal{M}_k$$
- $Q_k : \mathcal{M} \to \mathcal{M}_k$
$$(Q_k w, v_k) = (w, v_k) \qquad \forall v_k \in \mathcal{M}_k$$
- $A_k : \mathcal{M}_k \to \mathcal{M}_k$
$$(A_k w_k, v_k) = A(w_k, v_k) \qquad \forall v_k \in \mathcal{M}_k$$

$(A := A_J)$

Orthogonal decomposition

$$\mathcal{O}_k := \{ \phi \mid \phi = (Q_k - Q_{k-1}) \psi, \ \psi \in \mathcal{M} \} , \qquad Q_0 := 0$$

Then

$$\mathcal{M} = \mathcal{O}_1 + \dots + \mathcal{O}_J$$

From the definition of operators

$$Q_t A = A_t P_t$$

$$Q_t Q_l = Q_l Q_t = Q_l \quad \text{for } l \le t$$

From the last equality

$$(Q_t - Q_{t-1})(Q_l - Q_{l-1}) = 0 \qquad \text{for } l \neq t$$

$\Rightarrow$ $(v_l, v_t) = 0$ for $v_l \in V_l$, $v_t \in V_t$ $\quad l \neq t$

$\Rightarrow$ decomposition of M into $V_1 + \dots + V_J$ is orthogonal

<u>First preconditioner</u>

$$\cancel{B_t = \sum_{t=1}^{J} \lambda_t^{-1} A (Q_t - Q_{t-1}) A v}$$

$$B = \sum_{t=1}^{J} \lambda_t^{-1} (Q_t - Q_{t-1}),$$

where λ_t is the spectral radius of A_t.

Then B is symmetric and positive definite and

$$A(BAv, v) = \sum_{t=1}^{J} \lambda_t^{-1} \|(Q_t - Q_{t-1}) Av\|^2 \qquad v \in M$$

$$A(BAv, v) = A\Big(\sum_{t=1}^{J} \lambda_t^{-1} (Q_t - Q_{t-1}) Av, v\Big) =$$
$$= \sum_{t=1}^{J} \lambda_t^{-1} \, A\big((Q_t - Q_{t-1}) Av, v\big) =$$
$$= \sum \lambda_t^{-1} \big((Q_t - Q_{t-1}) Av, Av\big)$$
$$= \sum \lambda_t^{-1} \big((Q_t - Q_{t-1}) Av, (Q_t - Q_{t-1}) Av\big) \qquad \text{since } Q_t \text{ is projection}$$

Then

$$A(BAv, v) \le \sum_{t=1}^{J} \lambda_t^{-1} \|Q_t A v\|^2 \le \sum_{t=1}^{J} A(P_t v, P_t v)$$

$$\lambda_t^{-1}(Q_t A v, Q_t A v) = \lambda_t^{-1} A(P_t v, Q_t A v) \le 1 \cdot (P_t v, A_t P_t v)$$
$$= A(P_t v, P_t v) \quad \checkmark$$

$$(Q_t v - Q_{t-1} v, Q_t v - Q_{t-1} v) = \|Q_t v\|^2 + \|Q_{t-1} v\|^2 \underbrace{- 2(Q_t v, Q_{t-1} v)}_{Q_{t-1} Q_t = Q_{t-1}}$$

def Q_{t-1}

$$= -2(Q_{t-1} Q_t v, Q_{t-1} v) = -2(Q_{t-1} v, Q_{t-1} v) = -2\|Q_{t-1} v\|^2$$

$$\Rightarrow \|(Q_t - Q_{t-1}) v\|^2 = \|Q_t v\|^2 - \|Q_{t-1} v\|^2 \le \|Q_t v\|^2$$

from which $A(BAv, v) \leq J \cdot A(v, v) \quad \forall v \in M$

$\|P_k\|_A \leq 1$ is used here in the proof

assumption: for $k = 1, \ldots, J$ there exists $C_1 > 0$ such that

$$(A.1) \qquad \|(I - Q_{k-1})v\|^2 \leq C_1 \lambda_k^{-1} A(v, v) \quad \forall v \in M$$

THM 1: Assume (A.1). Then

proved above

$$C_1^{-1} J^{-1} A(v,v) \leq A(BAv, v) \leq J A(v,v) \qquad \forall v \in M$$

Proof (of the lower bound)

$$A(v,v) = \sum_{k=1}^{J} A((Q_k - Q_{k-1})v, v)$$

$$= \sum_{k=1}^{J} (v, (Q_k - Q_{k-1})Av)$$

$(Q_k - Q_{k-1})$ is a projection

$$= \sum_{k=1}^{J} ((Q_k - Q_{k-1})v, (Q_k - Q_{k-1})Av)$$

$rhs \in M_k$ and from definition of Q_k

$$= \sum_{k=1}^{J} ((I - Q_{k-1})v, (Q_k - Q_{k-1})Av)$$

By C-Schwarz inequality and (A.1)

$$A(v,v) \leq \sum_{k=1}^{J} \|(I - Q_{k-1})v\| \cdot \|(Q_k - Q_{k-1})Av\| \leq$$

$$\leq C_1^{+1/2} \sum_{k=1}^{J} \lambda_k^{-1/2} A(v,v) \cdot \|(Q_k - Q_{k-1})Av\|$$

Therefore $A(v,v)^{1/2} \leq C_1^{+1/2} J^{1/2} (A(BAv, v))^{1/2}$ $\square$

The form of the preconditioner $B = \sum \lambda_k^{-1}(Q_k - Q_{k-1})$ is not very efficient for computations (two projections in each summand).

Write B therefore as

$$B = \sum_{\ell=1}^{J-1} (\lambda_\ell^{-1} - \lambda_{\ell+1}^{-1}) Q_\ell \qquad + \lambda_J^{-1} I$$

This motivates the <u>second preconditioner</u>

$$((*)) \qquad \hat{B} = \sum_{\ell=1}^{J} \lambda_\ell^{-1} Q_\ell$$

If $\lambda_{\ell+1} \geq \sigma \lambda_\ell$ for some $\sigma > 1$ and all ℓ, then

$$(1-\sigma^{-1})(\hat{B}u, u) \leq (Bu, u) \leq (\hat{B}u, u) \quad \forall u \in M$$

Paper describes also a more general choice

$$B = \sum_{\ell=1}^{J} R_\ell Q_\ell$$

for some symmetric pos.-definite operators R_ℓ (smoothers), but we will stick with $(*)$, i.e. $R_\ell = \lambda_\ell^{-1} I$

B with $R_\ell = \lambda_\ell^{-1} I$ satisfies (A.2) with $C_2 = C_3 = 1$

If $\{\psi_\ell^{\,l}\}$ is an <u>orthonormal</u> basis of M_ℓ. Then

$$\lambda_\ell^{-1} u = \lambda_\ell^{-1} \sum_l (u, \psi_\ell^{\,l}) \psi_\ell^{\,l} \qquad \forall u \in M_\ell$$

Moreover $\qquad R_\ell Q_\ell u = \lambda_\ell^{-1} \sum_l (u, \psi_\ell^{\,l}) \psi_\ell^{\,l}$

so that application of Q_ℓ (L^2 projection) is computable without solving a system with the global Gram matrix.

An orthonormal FEM basis is typically not available.
However, the rest of the paper shows, that under some assumptions
one can consider the preconditioner

$$\mathcal{B}v = \sum_{k=1}^{J} \sum_{\ell} (v, \phi_k^\ell)\, \phi_k^\ell$$

even for the nodal (standard) FEM basis.

-> this ~~basis~~ preconditioner can be applied in parallel on all levels at once
(in contrast to MG V-cycle)
-> the condition number of the prec. problem depends on J
(or J^2 in less regular problems)