On the Algebraic Error in Numerical Solution of Partial Differential Equations
Part II – Estimating algebraic error using flux reconstructions

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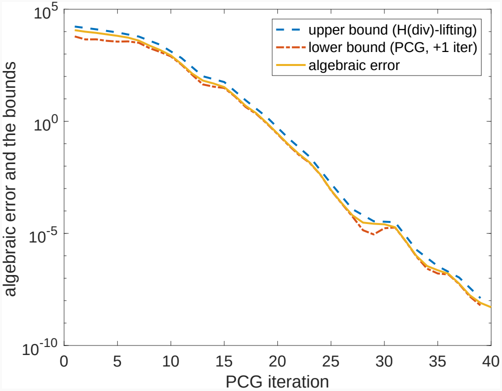
upper bound (H(div)-lifting)
lower bound (PCG, +1 iter)
algebraic error
algebraic error and the bounds
PCG iteration
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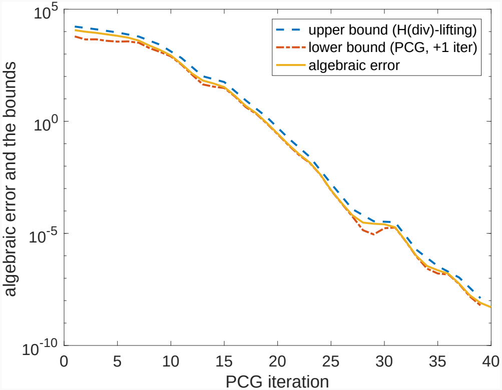

algebraic error and the bounds
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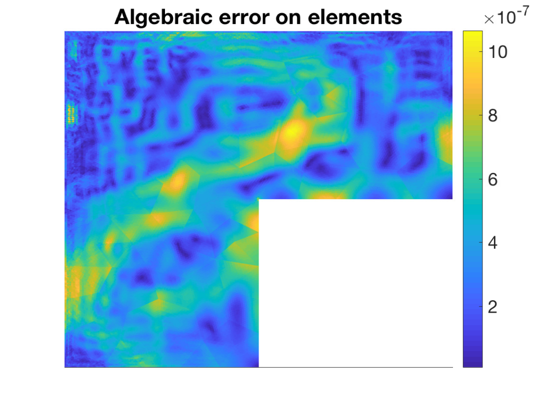
algebraic error



Algebraic error on elements

Outline

Introduction and notation

Upper bound on total error and quasi-equilibrated flux reconstruction

Three algebraic error upper bounds

Construction of fluxes

Numerical results

Setting and notation

Poisson problem: $-\operatorname{div}(\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$

Weak solution $u \in V \equiv H^1_0(\Omega)$,

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in V\,,$$

flux $\boldsymbol{\sigma} \equiv -\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$, $\operatorname{div} \boldsymbol{\sigma} = f$.

FEM discrete approximation $u_h \in V_h \subset V$,

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h\,.$$

Algebraic problem, using the basis $\Phi = \{\phi_1, \dots, \phi_N\}$ of V_h,

$$\mathbf{A}\mathsf{U} = \mathsf{F}, \qquad (\mathbf{A})_{j\ell} = (\nabla \phi_\ell, \nabla \phi_j), \quad \mathsf{F}_j = (f, \phi_j), \quad u_h = \Phi \mathsf{U}.$$

Inexact iterative **solution** $\mathsf{U}^i \approx \mathsf{U}$, $u_h^i = \Phi \mathsf{U}^i$,

residual $\mathsf{R}^i = \mathsf{F} - \mathbf{A}\mathsf{U}^i$.

Errors and error measure

$$\underbrace{u - u_h^i}_{\text{total error}} = \underbrace{u - u_h}_{\text{discretization error}} + \underbrace{u_h - u_h^i}_{\text{algebraic error}}$$

Energy norm of the total error

$$\|\nabla(u - u_h^i)\| = \sup_{v \in V, \|\nabla v\|=1} (\nabla(u - u_h^i), \nabla v) = \sup_{v \in V, \|\nabla v\|=1} (f, v) - (\nabla u_h^i, \nabla v)\,.$$

Energy norm of the algebraic error

$$\|\nabla(u - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v\|=1} (\nabla(u_h - u_h^i), \nabla v_h)\,.$$

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Energy norm of the algebraic error

$$\|\nabla(u - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v\|=1} (\nabla(u_h - u_h^i), \nabla v_h)\,.$$

Due to Galerkin orthogonality,

$$\|\nabla(u - u_h^i)\|^2 = \|\nabla(u - u_h)\|^2 + \|\nabla(u_h - u_h^i)\|^2\,.$$

Upper bound on the total error

Energy norm of the total error

$$\|\nabla(u - u_h^i)\| = \sup_{v \in V, \|\nabla v\| = 1} (\nabla(u - u_h^i), \nabla v)\,.$$

For any $\mathbf{d} \in \mathbf{H}(\mathrm{div}, \Omega)$,

$$\begin{aligned}
(\nabla(u - u_h^i), \nabla v) &= (f, v) - (\nabla u_h^i, \nabla v) \\
&= (f, v) + (\mathbf{d}, \nabla v) - (\mathbf{d}, \nabla v) - (\nabla u_h^i, \nabla v) \\
&= (f - \operatorname{div} \mathbf{d}, v) - (\nabla u_h^i + \mathbf{d}, \nabla v)\ .
\end{aligned}$$

Quasi-equilibrated flux reconstruction

We construct *representation* $r_h^i \in L^2(\Omega)$ of the algebraic residual R^i and the *approximate flux* $\mathbf{d}_h^i \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ such that

$$\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i \,.$$

Then

$$\begin{aligned}(\nabla(u - u_h^i), \nabla v) &= (f - \operatorname{div} \mathbf{d}_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v) \\ &= (f - f_h, v) + (r_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v)\,,\end{aligned}$$

[Prager, Synge (1947)]

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giving

$$\|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{osc}} + \sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v) + \|\nabla u_h^i + \mathbf{d}_h^i\| \,.$$

η_{osc}	data oscillation
$\sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v)$	$\rightarrow$ algebraic error estimate (bound)
$\|\nabla u_h^i + \mathbf{d}_h^i\|$	discretization error indicator

Residual representation

How to construct the representation $r_h^i \in L^2(\Omega)$?

If r_h^i is such that

$$(r_h^i, \phi_j) = \mathrm{R}_j^i \,, \quad j = 1, \dots, N \,,$$

where ϕ_j is a basis function of V_h and R_j^i is the associated element of R^i , we have for $v_h \in V_h$,

$$(\nabla(u_h - u_h^i), \nabla v_h) = (f, v_h) - (\nabla u_h^i, \nabla v_h) = (r_h^i, v_h) \,.$$

Then

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\| = 1} (r_h^i, v_h) \le \sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v) \,.$$

[Papež, Strakoš, Vohralík (2018)]

Bound 1: worst-case bound

Using Cauchy–Schwarz and Friedrichs inequalities

$$(r_h^i, v) \leq \|r_h^i\| \cdot \|v\| \leq \|r_h^i\| \cdot C_F h_\Omega \|\nabla v\|,$$

which gives

worst-case upper bounds:

$$\|\nabla(u - u_h^i)\| \leq \eta_{\text{osc}} + C_F h_\Omega \|r_h^i\| + \|\nabla u_h^i + \mathbf{d}_h^i\|$$

$$\|\nabla(u_h - u_h^i)\| \leq \qquad + C_F h_\Omega \|r_h^i\|$$

Residual representation and worst-case bound I

Construction of $r_h^i = \Phi \mathsf{C}^i \in V_h$ requires solution of

$$\mathbf{G}\mathsf{C}^i = \mathsf{R}^i\,, \qquad (\mathbf{G})_{j\ell} \equiv (\phi_\ell, \phi_j)\,.$$

Then

$$\|\mathsf{R}^i\|_{\mathbf{A}^{-1}} = \|\nabla(u_h - u_h^i)\| \leq C_F h_\Omega \|r_h^i\| = C_F h_\Omega \|\mathsf{R}^i\|_{\mathbf{G}^{-1}}$$

holds for any prescribed R^i. Considering the attainable bound

$$\begin{aligned}\|\mathsf{R}^i\|^2_{\mathbf{A}^{-1}} &= (\mathsf{R}^i, \mathbf{A}^{-1}\mathsf{R}^i) = (\mathbf{G}^{-1/2}\mathsf{R}^i, \mathbf{G}^{1/2}\mathbf{A}^{-1}\mathbf{G}^{1/2}\mathbf{G}^{-1/2}\mathsf{R}^i) \\ &\leq \|\mathbf{G}^{1/2}\mathbf{A}^{-1}\mathbf{G}^{1/2}\| \cdot \|\mathbf{G}^{-1/2}\mathsf{R}^i\|^2 = \|\mathbf{G}^{1/2}\mathbf{A}^{-1}\mathbf{G}^{1/2}\| \cdot \|\mathsf{R}^i\|^2_{\mathbf{G}^{-1}}\end{aligned}$$

and R^i for which the equality is attained,

$$\|\mathbf{G}^{1/2}\mathbf{A}^{-1}\mathbf{G}^{1/2}\| \leq (C_F h_\Omega)^2\,.$$

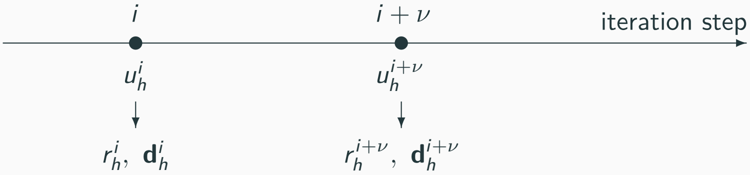
Residual representation and worst-case bound II

In order to avoid solution of the system with the mass matrix $\mathbf{G}$, we construct the algebraic residual representation $r_h^i \notin V_h$, piecewise discontinuous polynomial of degree of u_h, *locally* on each element.

The bound using $r_h^i \notin V_h$ constructed locally is weaker than the bound from the global construction, i.e.

$$\|\mathrm{R}^i\|_{\mathbf{A}^{-1}} \leq C_F h_\Omega \|\mathrm{R}^i\|_{\mathbf{G}^{-1}} \leq C_F h_\Omega \|r_h^i\| \,.$$

Bound 2: Additional iteration steps



Flux reconstruction in i-th iteration, r_h^i is the representation of R^i,

$$\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i\,,$$

in $(i+\nu)$-th iteration, $r_h^{i+\nu}$ is the representation of $\mathrm{R}^{i+\nu}$,

$$\operatorname{div} \mathbf{d}_h^{i+\nu} = f_h - r_h^{i+\nu}\,.$$

Then

$$r_h^i = -\operatorname{div} \mathbf{d}_h^i + \operatorname{div} \mathbf{d}_h^{i+\nu} + r_h^{i+\nu}\,.$$

[Ern, Vohralík (2013)]

Bound 2: Upper bounds using additional iterations

Then

$$(\nabla(u - u_h^i), \nabla v) = (f - f_h, v) + (r_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v)$$
$$= (f - f_h, v) + (\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}, \nabla v) + (r_h^{i+\nu}, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v)\,,$$

and

$$(\nabla(u_h - u_h^i), \nabla v_h) = (\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}, \nabla v_h) + (r_h^{i+\nu}, v_h)\,.$$

Upper bounds: [Papež, Strakoš, Vohralík (2018)]

$$\|\nabla(u - u_h^i)\| \le \eta_{\mathrm{osc}} + \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\| + \|\nabla u_h^i + \mathbf{d}_h^i\|$$
$$\|\nabla(u_h - u_h^i)\| \le \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\|$$

crucial question: How to choose ν?

Bound 2: Upper bounds using additional iterations

Recall:

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\|$$

Crucial question: How to choose ν?

Bound 2: Upper bounds using additional iterations

Recall:

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\|$$

Crucial question: How to choose ν?

Idea: compare the terms $\|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\|$ and $C_F h_\Omega \|r_h^{i+\nu}\|$.

Bound 2: Upper bounds using additional iterations

Recall:

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\|$$

Crucial question: How to choose ν?

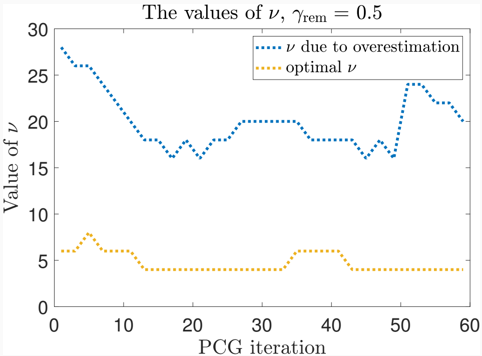
Idea: compare the terms $\|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\|$ and $C_F h_\Omega \|r_h^{i+\nu}\|$.

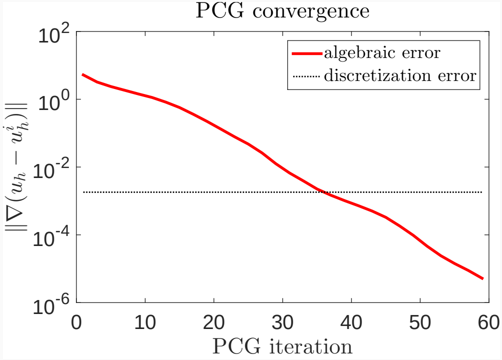
Find the smallest ν such that

$$\gamma_{\mathrm{rem}} \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| \geq C_F h_\Omega \|r_h^{i+\nu}\|,$$

where γ_{rem} can be set, for example, as $\gamma_{\mathrm{rem}} = 0.5$.

Bound 2: Upper bounds using additional iterations





Comparison of the number ν of additional iterations; the optimal number (yellow) and the number due to the overestimation in the worst-case bound (blue)

For this price, we get an upper bound with efficiency close to $1.5 = 1 + \gamma_{\rm rem}$.

Bound 3: construction of an algebraic flux

Constructing $\mathbf{a}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$ such that

$$\operatorname{div} \mathbf{a}_h^i = r_h^i ,$$

we have

$$(r_h^i, v) = (\operatorname{div} \mathbf{a}_h^i, v) = -(\mathbf{a}_h^i, \nabla v)$$

giving

$$\sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v) \leq \|\mathbf{a}_h^i\| .$$

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$$\sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v) \leq \|\mathbf{a}_h^i\| .$$

Then

Bounds based on algebraic flux: [Papež, Rüde, Vohralík, Wohlmuth (2020)]

$$\|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{osc}} + \|\mathbf{a}_h^i\| + \|\nabla u_h^i + \mathbf{d}_h^i\|$$

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{a}_h^i\|$$

Construction of fluxes

In this part:

- a subspace of $\mathbf{H}(\mathrm{div}, \Omega)$: Raviart–Thomas(–Nédélec) space
- flux reconstruction $\mathbf{d}_h^i$, $\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i$

[Braess, Schöberl (2008)]
[Ern, Vohralík (2013)]

- algebraic flux $\mathbf{a}_h^i$, $\operatorname{div} \mathbf{a}_h^i = r_h^i$

[Papež, Rüde, Vohralík, Wohlmuth (2020)]
[Papež, Vohralík (2021?)]

Raviart–Thomas(–Nédélec) functions

For flux (re)constructions we use RTN space

$$\mathbf{RTN}_q(K) = \left\{\mathbf{v} \in \left[\mathbb{P}_q(K)\right]^d + \mathbb{P}_q(K)\mathbf{x}\right\} \subset \mathbf{H}(\mathrm{div}, \Omega)\,.$$

We set $q := p$, where p is the degree of FEM approximation. To prove the global and local efficiency of upper bound on the total error, we need $q := p + 1$.

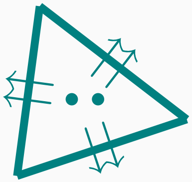

Illustration of degrees of freedom for $q = 1$. In 2D, $\dim(\mathbf{RTN}_q(K)) = (q+1)(q+3)$.

Quasi-equilibrated flux

Goal: construct $\mathbf{d}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{d}_h^i|_K \in \mathbf{RTN}_q(K)$, $\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i$.

To obtain a tight bound, we should minimize the error indicator $\|\nabla u_h + \mathbf{d}_h^i\|$.

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To obtain a tight bound, we should minimize the error indicator $\|\nabla u_h + \mathbf{d}_h^i\|$.

Local construction based on a partition of unity by piecewise affine *hat functions* corresponding to each vertex of the mesh,

$$\mathbf{d}_h^i := \sum_{\mathrm{a} \in \mathcal{V}_h} \mathbf{d}_{h,\mathrm{a}}^i \qquad \mathrm{div}\, \mathbf{d}_{h,\mathrm{a}}^i = \psi_{h,\mathrm{a}}(f_h - r_h^i), \qquad \|\psi_{h,\mathrm{a}} \nabla u_h^i + \mathbf{d}_{h,\mathrm{a}}^i\| \to \min .$$

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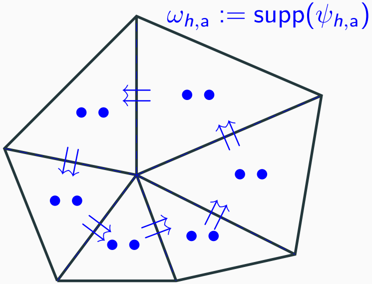

Degrees of freedom of $\mathbf{d}_h^i$ for $q = 1$.

Quasi-equilibrated flux

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The local flux $\mathbf{d}_{h,\mathrm{a}}^i$ is given as the solution of

$$\begin{aligned} (\mathbf{d}_{h,\mathrm{a}}^i, \mathbf{v}_h)_{\omega_{h,\mathrm{a}}} - (\gamma_h^{\mathrm{a}}, \operatorname{div} \mathbf{v}_h)_{\omega_{h,\mathrm{a}}} &= -(\psi_{h,\mathrm{a}} \nabla u_h^i, \mathbf{v}_h)_{\omega_{h,\mathrm{a}}} \\ (\operatorname{div} \mathbf{d}_{h,\mathrm{a}}^i, q_h)_{\omega_{h,\mathrm{a}}} &= (f \psi_{h,\mathrm{a}} - \nabla u_h^i \cdot \nabla \psi_{h,\mathrm{a}} - r_h \psi_{h,\mathrm{a}}, q_h)_{\omega_{h,\mathrm{a}}} . \end{aligned}$$

$\mathbf{d}_{h,\mathrm{a}}^i$, $\mathbf{v}_h$ elementwise $\mathbf{RTN}_q$ functions from $\mathbf{H}(\mathrm{div}, \Omega)$ with no flux through $\partial \omega_{h,\mathrm{a}}$

γ_h^{a}, q_h elementwise q-order polynomials, discontinuous, with zero mean on $\omega_{h,\mathrm{a}}$

Quasi-equilibrated flux

Goal: construct $\mathbf{d}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{d}_h^i|_K \in \mathbf{RTN}_q(K)$, $\mathrm{div}\, \mathbf{d}_h^i = f_h - r_h^i$.

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Thanks to the compatibility condition

$$(f \psi_{h,\mathrm{a}} - \nabla u_h^i \cdot \nabla \psi_{h,\mathrm{a}} - r_h \psi_{h,\mathrm{a}}, 1)_{\omega_{h,\mathrm{a}}} = 0,$$

we have $\mathrm{div}\, \mathbf{d}_h^i = f_h - r_h^i$.

Algebraic flux

Goal: construct $\mathbf{a}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{a}_h^i|_K \in \mathbf{RTN}_q(K)$, $\operatorname{div} \mathbf{a}_h^i = r_h^i$.

Can be generalized to construct a flux with an arbitrary (prescribed) divergence.

Algebraic flux

Goal: construct $\mathbf{a}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{a}_h^i|_K \in \mathbf{RTN}_q(K)$, $\operatorname{div} \mathbf{a}_h^i = r_h^i$.

Crucial difficulty: there is no compatibility condition as above for $\mathbf{d}_h^i$!

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Idea: use a solution on a coarser mesh $\rightarrow$ multilevel construction

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Crucial difficulty: there is no compatibility condition as above for $\mathbf{d}_h^i$!

Idea: use a solution on a coarser mesh → multilevel construction

Two-level construction

Step 1: Solve for piecewise affine scalar function on the coarser level

$$(\nabla \rho_{0,\mathrm{alg}}, \nabla v_0) = (r_h^i, v_0) \qquad \forall v_0 \in \mathbb{P}_1(\mathcal{T}_H).$$

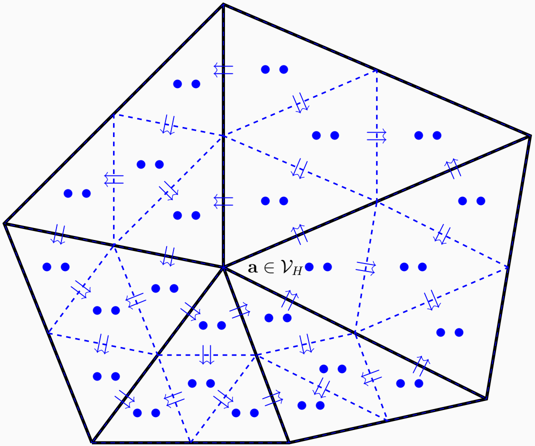
Step 2: On the finer level

$$\begin{aligned}(\mathbf{a}_{h,\mathrm{a}}^i, \mathbf{v}_h)_{\omega_{H,\mathrm{a}}} - (\gamma_h^{\mathrm{a}}, \operatorname{div} \mathbf{v}_h)_{\omega_{H,\mathrm{a}}} &= 0 \\ (\operatorname{div} \mathbf{a}_{h,\mathrm{a}}^i, q_h)_{\omega_{H,\mathrm{a}}} &= (r_h^i \psi_{H,\mathrm{a}} - \nabla \rho_{0,\mathrm{alg}} \cdot \nabla \psi_{H,\mathrm{a}}, q_h)_{\omega_{H,\mathrm{a}}}.\end{aligned}$$

Step 3: Define

$$\mathbf{a}_h^i := \sum_{\mathrm{a} \in \mathcal{V}_H} \mathbf{a}_{h,\mathrm{a}}^i.$$

Algebraic flux – two-level construction



coarse patch $\omega_{H,\mathrm{a}}$ for $\mathrm{a} \in \mathcal{V}_H$ (full line)

fine mesh $\mathcal{T}_h$ of $\omega_{H,\mathrm{a}}$ (dashed line)

degrees of freedom for $\mathbf{a}^i_{h,\mathrm{a}}$ for $q = 1$ (arrows and bullets)

Algebraic flux – multilevel construction

In multilevel setting, we have

$$\mathbf{a}_h^i := \sum_{j=1}^{J} \sum_{\mathrm{a}\in\mathcal{V}_{j-1}} \mathbf{a}_{j,\mathrm{a}}^i,$$

where $\mathbf{a}_{1,\mathrm{a}}^i$ are defined analogously to two-level setting and, for $1 < j \leq J$,

$$\begin{aligned}(\mathbf{a}_{j,\mathrm{a}}^i, \mathbf{v}_j)_{\omega_{j-1,\mathrm{a}}} - (\gamma_j^{\mathrm{a}}, \operatorname{div} \mathbf{v}_j)_{\omega_{j-1,\mathrm{a}}} &= 0 \\ (\operatorname{div} \mathbf{a}_{j,\mathrm{a}}^i, q_j)_{\omega_{j-1,\mathrm{a}}} &= ((\mathrm{Id} - \Pi_{j-1}^q)(r_h^i \psi_{H,\mathrm{a}}), q_j)_{\omega_{j-1,\mathrm{a}}}.\end{aligned}$$

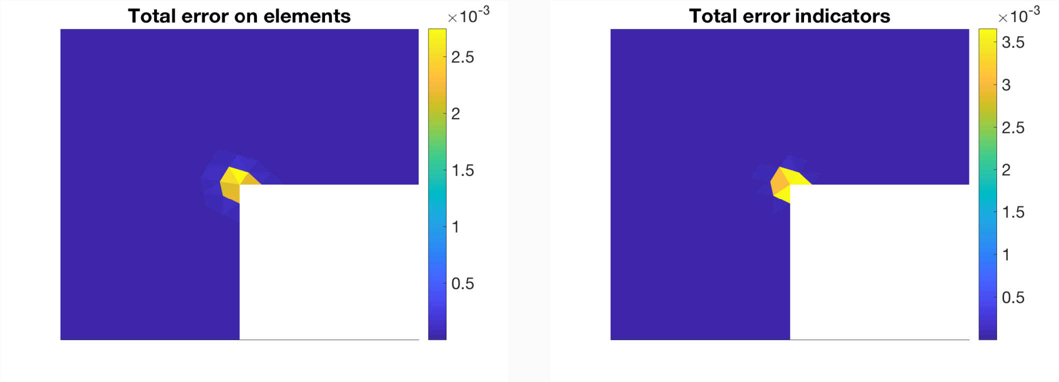


Figure 1: L-shape problem, $p = 3$: elementwise distribution of the total energy error $\|\nabla(u - u_h^i)\|_K$ (left) and of the local error indicators (right) after 28 PCG iterations. We plot in both figures the part $[-0.1, 0.1] \times [-0.1, 0.1]$ of the discretization domain Ω

taken from [Papež, Rüde, Vohralík, Wohlmuth (2020)]

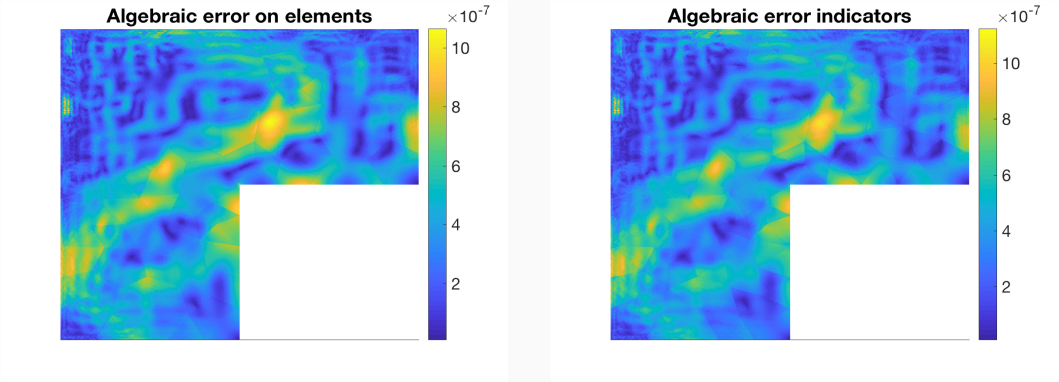


Figure 2: L-shape problem, $p = 3$: elementwise distribution of the algebraic energy error $\|\nabla(u_h - u_h^i)\|_K$ (left) and of the local error indicators (right) after 28 PCG iterations

Numerical results

p	PCG iter	algebraic error	eff. index UB	eff. index LB	total error	eff. index UB	eff. index LB	discretization error	eff. index UB	eff. index LB
1	4	8.9×10^{-2}	1.02	1.00^{-1}	9.1×10^{-2}	1.26	1.03^{-1}	2.2×10^{-2}	3.35	—
	8	3.8×10^{-4}	1.01	1.00^{-1}	2.2×10^{-2}	1.22	1.12^{-1}		1.22	1.12^{-1}
2	4	6.2×10^{-1}	1.01	1.00^{-1}	6.2×10^{-1}	1.07	1.00^{-1}	8.9×10^{-3}	2.61×10^{1}	—
	8	6.0×10^{-3}	1.01	1.00^{-1}	1.1×10^{-2}	1.65	1.58^{-1}		1.88	2.86^{-1}
	12	1.9×10^{-4}	1.01	1.00^{-1}	8.9×10^{-3}	1.33	1.28^{-1}		1.33	1.28^{-1}
3	7	1.0	1.00	1.00^{-1}	1.0	1.05	1.00^{-1}	5.3×10^{-3}	6.29×10^{1}	—
	14	3.1×10^{-2}	1.01	1.00^{-1}	3.1×10^{-2}	1.24	1.01^{-1}		4.48	—
	21	1.7×10^{-3}	1.00	1.00^{-1}	5.6×10^{-3}	1.68	1.48^{-1}		1.74	1.59^{-1}
	28	9.6×10^{-5}	1.00	1.00^{-1}	5.3×10^{-3}	1.46	1.41^{-1}		1.46	1.41^{-1}
4	7	1.2	1.01	1.00^{-1}	1.2	1.08	1.00^{-1}	3.8×10^{-3}	1.30×10^{2}	—
	14	5.0×10^{-2}	1.01	1.00^{-1}	5.1×10^{-2}	1.14	1.00^{-1}		7.34	—
	21	3.4×10^{-3}	1.00	1.00^{-1}	5.0×10^{-3}	1.77	1.50^{-1}		2.19	—
	28	1.8×10^{-4}	1.01	1.00^{-1}	3.8×10^{-3}	1.52	1.60^{-1}		1.52	1.60^{-1}

L-shape problem, PCG solver: effectivity of the error bounds

Numerical results

	MG	algebraic	eff. index		total	eff. index		discretization	eff. index	
p	iter	error	UB	LB	error	UB	LB	error	UB	LB
1	1	1.4	1.14	1.03^{-1}	1.4	1.61	1.03^{-1}	2.2×10^{-2}	8.31×10^{1}	—
	2	6.7×10^{-2}	1.14	1.04^{-1}	7.0×10^{-2}	1.61	1.10^{-1}		4.22	—
	3	4.3×10^{-3}	1.16	1.07^{-1}	2.3×10^{-2}	1.37	1.16^{-1}		1.38	1.17^{-1}
	4	4.1×10^{-4}	1.17	1.09^{-1}	2.2×10^{-2}	1.22	1.13^{-1}		1.22	1.13^{-1}
2	1	2.6	1.19	1.01^{-1}	2.6	1.78	1.01^{-1}	8.9×10^{-3}	4.31×10^{2}	—
	2	8.9×10^{-2}	1.19	1.01^{-1}	8.9×10^{-2}	1.79	1.01^{-1}		1.49×10^{1}	—
	3	2.2×10^{-3}	1.18	1.01^{-1}	9.2×10^{-3}	1.55	1.42^{-1}		1.58	1.50^{-1}
	4	8.6×10^{-5}	1.19	1.02^{-1}	8.9×10^{-3}	1.32	1.29^{-1}		1.32	1.29^{-1}
3	1	2.4	1.19	1.00^{-1}	2.4	1.72	1.00^{-1}	5.3×10^{-3}	6.29×10^{2}	—
	2	1.1×10^{-1}	1.20	1.00^{-1}	1.1×10^{-1}	1.76	1.00^{-1}		2.92×10^{1}	—
	3	3.6×10^{-3}	1.18	1.00^{-1}	6.4×10^{-3}	1.89	1.47^{-1}		2.19	6.44^{-1}
	4	1.8×10^{-4}	1.17	1.01^{-1}	5.3×10^{-3}	1.48	1.42^{-1}		1.48	1.42^{-1}
4	1	2.6	1.18	1.00^{-1}	2.6	1.68	1.00^{-1}	3.8×10^{-3}	9.43×10^{2}	—
	2	1.3×10^{-1}	1.18	1.00^{-1}	1.3×10^{-1}	1.71	1.00^{-1}		4.93×10^{1}	—
	3	6.0×10^{-3}	1.16	1.00^{-1}	7.1×10^{-3}	1.87	1.18^{-1}		3.13	—
	4	3.5×10^{-4}	1.13	1.00^{-1}	3.8×10^{-3}	1.57	1.66^{-1}		1.57	1.67^{-1}

L-shape problem, multigrid V-cycle solver: effectivity of the error bounds

Algebraic flux – simplifications of multilevel construction

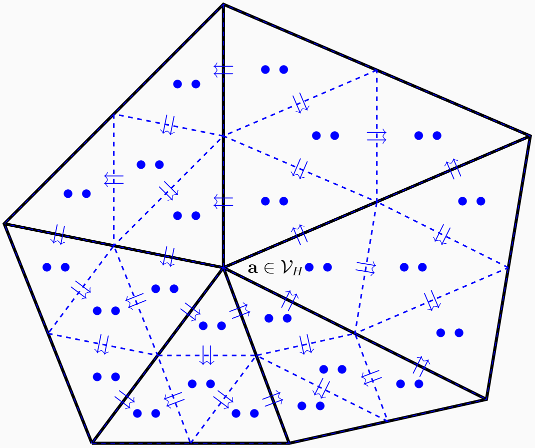
To reduce computational cost

- construct $\mathbf{a}_j^i \in \mathbf{RTN}_1(\mathcal{T}_j)$ for $1 \leq j < J$,
- on each "large" patch, replace the RTN solve by one $\mathbb{P}_1$-solve and several RTN solves on "small" patches
- as above but additionally replace RTN solves on "small" patches by a sweep over elements of the patch

However, this does not yield the same flux! The estimate may be (bit) worse.

[Papež, Vohralík (2021?)]

Algebraic flux – original "large" patch RTN solve

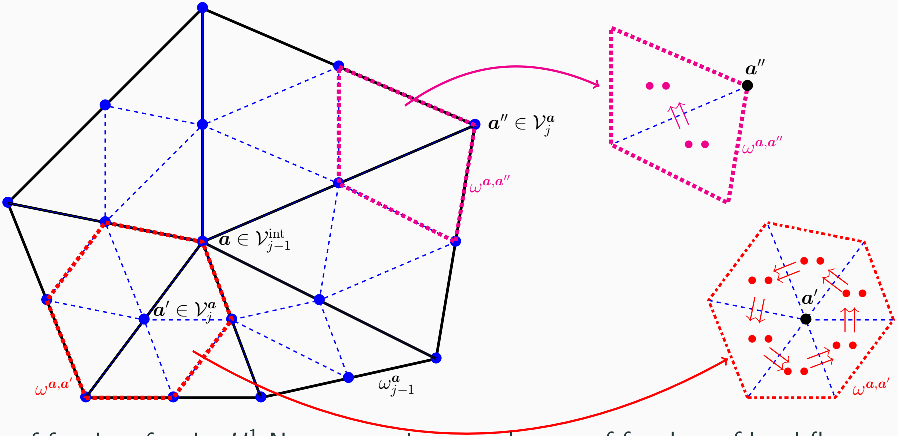


coarse patch $\omega_{H,\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_H$ (full line)

fine mesh $\mathcal{T}_h$ of $\omega_{H,\mathbf{a}}$ (dashed line)

degrees of freedom for $\mathbf{a}^i_{h,\mathbf{a}}$ for $q = 1$ (arrows and bullets)

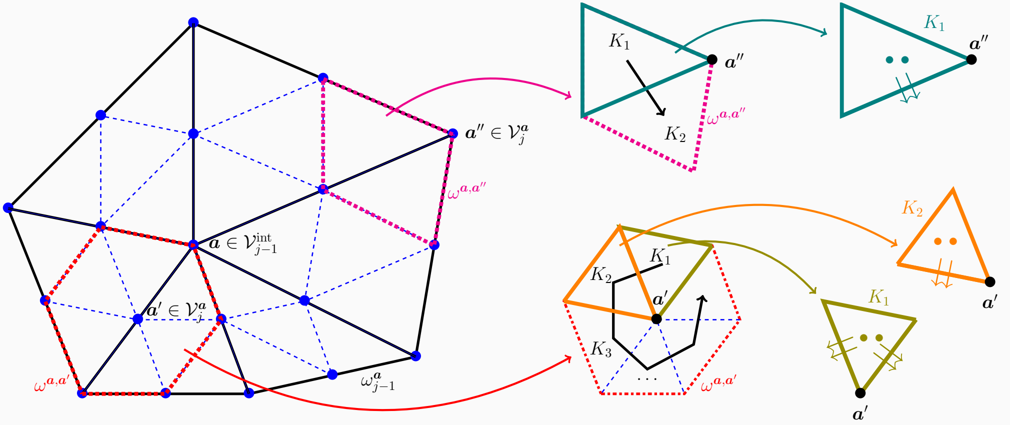
Algebraic flux – simplification 2



degrees of freedom for the H^1 Neumann solve (bullets)

degrees of freedom of local fluxes on small patches

Algebraic flux – simplification 3



degrees of freedom for the H^1 Neumann solve (bullets)

explicit run through small patch

degrees of freedom of local fluxes on single elements

Numerical results - simplifications, effectivity

	PCG	algebraic	eff. index			total	eff. index		
p	iter	error	orig.	simpl.1	simpl.2	error	orig.	simpl.1	simpl.2
1 (2.5×10^4)	4	8.9×10^{-2}	1.02	1.05	1.18	9.1×10^{-2}	1.26	1.29	1.42
	8	3.8×10^{-4}	1.01	1.03	1.17	2.2×10^{-2}	1.22	1.22	1.22
2 (1.0×10^5)	4	6.2×10^{-1}	1.01	1.03	1.18	6.2×10^{-1}	1.07	1.09	1.24
	8	6.0×10^{-3}	1.01	1.04	1.19	1.1×10^{-2}	1.65	1.67	1.75
	12	1.9×10^{-4}	1.01	1.03	1.18	8.9×10^{-3}	1.33	1.33	1.33
3 (2.3×10^5)	7	1.0	1.00	1.03	1.17	1.0	1.05	1.07	1.22
	14	3.1×10^{-2}	1.01	1.04	1.19	3.1×10^{-2}	1.24	1.27	1.42
	21	1.7×10^{-3}	1.00	1.03	1.15	5.6×10^{-3}	1.68	1.69	1.72
	28	9.6×10^{-5}	1.00	1.03	1.18	5.3×10^{-3}	1.46	1.46	1.46
4 (4.0×10^5)	7	1.2	1.01	1.02	1.17	1.2	1.08	1.10	1.25
	14	5.0×10^{-2}	1.01	1.04	1.18	5.1×10^{-2}	1.14	1.17	1.31
	21	3.4×10^{-3}	1.00	1.03	1.16	5.0×10^{-3}	1.77	1.78	1.87
	28	1.8×10^{-4}	1.01	1.04	1.18	3.8×10^{-3}	1.52	1.52	1.53

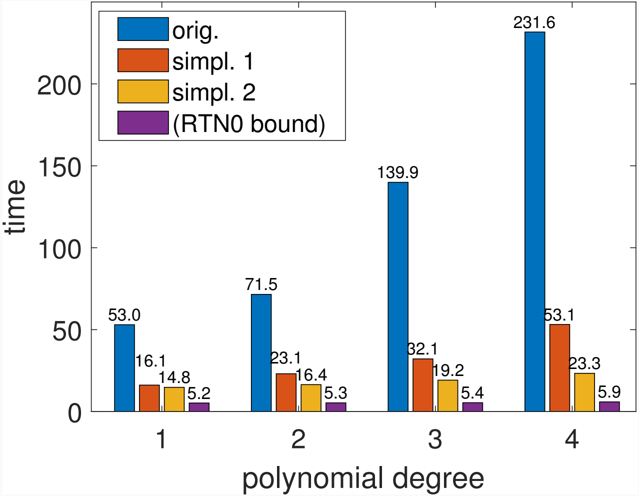
L-shape problem, PCG solver: effectivity of the error bounds with original and two cheaper constructions

Numerical results - simplifications, effectivity

p	MG iter	algebraic error	eff. index orig.	eff. index simpl.1	eff. index simpl.2	total error	eff. index orig.	eff. index simpl.1	eff. index simpl.2
1 (2.5×10^4)	1	1.4	1.14	1.18	1.37	1.4	1.60	1.64	1.83
	2	6.7×10^{-2}	1.14	1.19	1.38	7.0×10^{-2}	1.61	1.65	1.84
	3	4.3×10^{-3}	1.16	1.25	1.59	2.3×10^{-2}	1.37	1.39	1.45
	4	4.1×10^{-4}	1.17	1.31	1.76	2.2×10^{-2}	1.22	1.22	1.23
2 (1.0×10^5)	1	2.6	1.19	1.22	1.74	2.6	1.78	1.81	2.33
	2	8.9×10^{-2}	1.19	1.20	1.64	8.9×10^{-2}	1.79	1.80	2.24
	3	2.2×10^{-3}	1.18	1.21	1.64	9.2×10^{-3}	1.55	1.56	1.66
	4	8.6×10^{-5}	1.19	1.25	1.70	8.9×10^{-3}	1.32	1.32	1.32
3 (2.3×10^5)	1	2.4	1.19	1.20	1.59	2.4	1.72	1.74	2.12
	2	1.1×10^{-1}	1.20	1.19	1.59	1.1×10^{-1}	1.76	1.76	2.16
	3	3.6×10^{-3}	1.18	1.17	1.61	6.4×10^{-3}	1.89	1.88	2.13
	4	1.8×10^{-4}	1.17	1.16	1.66	5.3×10^{-3}	1.48	1.48	1.49
4 (4.0×10^5)	1	2.6	1.18	1.25	1.61	2.6	1.68	1.75	2.11
	2	1.3×10^{-1}	1.18	1.18	1.50	1.3×10^{-1}	1.71	1.72	2.03
	3	6.0×10^{-3}	1.16	1.15	1.46	7.1×10^{-3}	1.87	1.87	2.12
	4	3.5×10^{-4}	1.13	1.13	1.44	3.8×10^{-3}	1.57	1.57	1.60

L-shape problem, multigrid V-cycle solver: effectivity of the error bounds with original and two cheaper constructions

Numerical results - simplifications, timing



Timing of constructions for varying polynomial degree

Conclusion

For error estimators based on flux reconstructions

Positives

- Upper bounds on the errors without any unknown constants
- We can prove the efficiency of the estimators (global+local for total error estimators; global for algebraic error estimators)
- Easily parallelizable construction
- Technique can be applied to more problems

Drawbacks

- Very high computational cost
- Requires implementation of p-order RTN functions
- Mesh hierarchy is required

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Thank you for your attention!

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